

C^∞ Functions on the Stone-Čech Compactification of the Integers

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Abstract

We construct an algebra $A = \ell^{\infty\infty}(\mathbb{Z})$ of smooth functions which is dense in the pointwise multiplication algebra $\ell^\infty(\mathbb{Z})$ of sup-norm bounded functions on the integers \mathbb{Z} . The algebra A properly contains the sum of the algebra $A_c = \ell_c^\infty(\mathbb{Z})$ and the ideal $\mathcal{S}(\mathbb{Z})$, where A_c is the algebra of finite linear combinations of projections in $\ell^\infty(\mathbb{Z})$ and $\mathcal{S}(\mathbb{Z})$ is the pointwise multiplication algebra of Schwartz functions. The algebra A is characterized as the set of functions whose “first derivatives” vanish rapidly at each point in the Stone-Čech compactification of \mathbb{Z} .

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1 Introduction

In a previous paper [Sch, 1998], a notion of smooth functions on the Cantor set was developed. Recall that a *totally disconnected* topological space has a basis of clopen sets. The Cantor set is such a space. In this paper, we attempt to construct smooth functions on the Stone-Čech compactification of the integers. In addition to being totally disconnected, this space is *extremely*

disconnected, which means that the closure of every open set is clopen.

We will be working with the C^* -algebra of all bounded complex-valued functions (or sequences) on the integers \mathbb{Z} , under pointwise multiplication, with pointwise complex-conjugation for involution. We denote this algebra by $\ell^\infty(\mathbb{Z})$. None of the theorems will use the additive and multiplicative structure of \mathbb{Z} , so that any countable discrete set can be substituted for \mathbb{Z} . However, when specific objects are constructed in the examples, we may make reference to the underlying set of integers. Recall that the integers \mathbb{Z} consists of all whole numbers from $-\infty$ to ∞ , whereas the natural numbers \mathbb{N} contains only the whole numbers from 0 to ∞ . The norm on $\ell^\infty(\mathbb{Z})$ is the sup-norm $\|\cdot\|_\infty$, defined by $\|\varphi\|_\infty = \sup_{n \in \mathbb{Z}} |\varphi(n)|$.

We will use the notation $c_0(\mathbb{Z})$ for the complex-valued functions (sequences) on \mathbb{Z} which vanish at infinity.

2 The Stone-Čech Compactification of \mathbb{Z}

We recall the standard definition of the Stone-Čech compactification from [Roy, 1968]. Let \mathbf{F} be the set of all real-valued functions from \mathbb{Z} into the closed interval $I = [-1, 1]$. (So \mathbf{F} is the set of real-valued functions in the

unit ball of $\ell^\infty(\mathbb{Z})$.) Let

$$\mathbf{X} = \prod_{\mathbf{F}} I$$

be the \mathbf{F} -fold cartesian product of unit intervals I . By the Tychonoff theorem [Roy, 1968, Chapter 9, Theorem 19], this is a compact Hausdorff space. Let

$$i : \mathbb{Z} \hookrightarrow \mathbf{X} \quad \text{where} \quad i(n) \longmapsto \{f(n)\}_{f \in \mathbf{F}}$$

be the natural inclusion map.

Definition 1.1. We define the *Stone-Ćech Compactification of \mathbb{Z}* , denoted by $\beta(\mathbb{Z})$, to be the closure of the image $i(\mathbb{Z})$ in \mathbf{X} . We let $C(\beta(\mathbb{Z}))$ denote the continuous complex-valued functions on $\beta(\mathbb{Z})$. See [Roy, 1968] for the basic properties of $\beta(\mathbb{Z})$.

Definition 1.2. Let $n_0 \in \beta(\mathbb{Z})$, and let $\{n_\alpha\}_{\alpha \in \Lambda}$ be a net converging to n_0 , where Λ is some directed set, and $n_\alpha \in \mathbb{Z}$ for each $\alpha \in \Lambda$. Let $\varphi \in \ell^\infty(\mathbb{Z})$. Then $\varphi(n_0)$ is defined as the limit of the net $\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}$. In this way φ defines a continuous function on $\beta(\mathbb{Z})$, giving an isomorphism of commutative C^* -algebras $\ell^\infty(\mathbb{Z}) \cong C(\beta(\mathbb{Z}))$. (The fact that φ is continuous, and that the limit defining $\varphi(n_0)$ converges, can be verified directly from the definition of the Stone-Ćech compactification above, using the fact that the real and imaginary parts of $\varphi/\|\varphi\|_\infty$ are functions in \mathbf{F} .)

Let $p \in \ell^\infty(\mathbb{Z})$ be a projection. In other words for each $n \in \mathbb{Z}$, $p(n) = 0$ or $p(n) = 1$. Then p also defines a projection on $\beta(\mathbb{Z})$. In fact, if $n_0 \in \beta(\mathbb{Z})$ and $\{n_\alpha\}_{\alpha \in \Lambda}$ are as above, then $p(n_\alpha)$ is either eventually equal to 1 (in the case $p(n_0) = 1$) or eventually equal to 0 (in the case $p(n_0) = 0$). Thus each point $n_0 \in \beta(\mathbb{Z})$ defines a family \mathcal{F}_{n_0} of subsets of \mathbb{Z} , where $S \in \mathcal{F}_{n_0}$ if and only if the projection p whose support is equal to S satisfies $p(n_0) = 1$.

Definition 1.3. A family of subsets \mathcal{F} of \mathbb{Z} is a *filter* on \mathbb{Z} if it is closed under finite intersections

$$S, T \in \mathcal{F} \implies S \cap T \in \mathcal{F} \quad (1.4a)$$

and supersets

$$S \in \mathcal{F} \quad \text{and} \quad T \supset S \implies T \in \mathcal{F}, \quad (1.4b)$$

and if the empty set \emptyset is *not* in \mathcal{F} . A filter \mathcal{U} on \mathbb{Z} is an *ultrafilter* on \mathbb{Z} if

$$S \subseteq \mathbb{Z} \implies S \in \mathcal{F} \quad \text{or} \quad S^c \in \mathcal{F}. \quad (1.4c)$$

The map $n_0 \in \beta(\mathbb{Z}) \mapsto \mathcal{F}_{n_0}$ from the previous paragraph defines a map \mathcal{UF} from $\beta(\mathbb{Z})$ to the set of ultrafilters on \mathbb{Z} . (Use the definition of $\beta(\mathbb{Z})$ to check this.)

A *principal filter* is of the form $\mathcal{U}_n = \langle S \subseteq \mathbb{Z} \mid n \in S \rangle$ for some $n \in \mathbb{Z}$.

Such a filter is also an ultrafilter. The image of $\mathbb{Z} \subseteq \beta(\mathbb{Z})$ under the map \mathcal{UF}

is precisely the set of principal ultrafilters on \mathbb{Z} .

We show that the map \mathcal{UF} is an isomorphism by constructing an inverse map \mathcal{FU} . Let \mathcal{U} be any ultrafilter on \mathbb{Z} . Define a directed set Λ to be \mathcal{U} with superset order. That is $\alpha \leq \beta \iff \beta \subseteq \alpha$. Thus smaller sets are “bigger” in this order. In the case of the principal ultrafilter $\Lambda = \mathcal{U}_n$, the singleton $\{n\}$ is the biggest element. In general, Λ has a biggest element if and only if it comes from a principal ultrafilter. Next, for $\alpha \in \Lambda$ choose any $n_\alpha \in \alpha \subseteq \mathbb{Z}$. We show that $\{n_\alpha\}_{\alpha \in \Lambda}$ converges to an element of $\beta(\mathbb{Z})$.

Let $\varphi \in \mathbf{F}$, where \mathbf{F} is the defining family of functions for $\beta(\mathbb{Z})$ from Definition 1.1. We wish to show that $\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}$ converges. Define

$$\varphi_+(n) = \begin{cases} \varphi(n) & \text{if } \varphi(n) \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\varphi_-(n) = \begin{cases} -\varphi(n) & \text{if } \varphi(n) \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varphi = \varphi_+ - \varphi_-$, and $\varphi_+, \varphi_- \in \mathbf{F}$. It suffices to show that $\{\varphi_+(n_\alpha)\}_{\alpha \in \Lambda}$ and $\{\varphi_-(n_\alpha)\}_{\alpha \in \Lambda}$ each converge. So without loss of generality, we take φ with range in the unit interval $[0, 1]$.

If φ were a projection p , we would be done. Simply let S be the support of

p . If $S \in \mathcal{U} = \Lambda$, then $\{p(n_\alpha)\}_{\alpha \in \Lambda}$ is eventually 1 (when $\alpha \geq S$). Otherwise, it is eventually 0, and in either case they converge. We proceed by writing φ as an infinite series of projections. Define the set of integers

$$S_{\frac{1}{2}} = \{n \in \mathbb{Z} \mid \frac{1}{2} \leq \varphi(n) \leq 1\}.$$

Let $p_{\frac{1}{2}}$ be the projection corresponding to $S_{\frac{1}{2}}$. Then $\varphi - \frac{1}{2}p_{\frac{1}{2}}$ has its range in the interval $[0, \frac{1}{2}]$. We repeat the process to get a new function with range in the interval $[0, \frac{1}{4}]$, etc, until we get:

$$\varphi = \frac{1}{2}p_{\frac{1}{2}} + \frac{1}{4}p_{\frac{1}{4}} + \cdots = \sum_{q=1}^{\infty} \frac{1}{2^q}p_q, \quad (1.5)$$

an infinite series that converges absolutely (and geometrically fast) in sup norm to φ . To see that $\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}$ converges, now use a standard series argument. Let $\epsilon > 0$ be given, and find N large enough so that the sup norm of the tail

$$\left\| \sum_{q=N}^{\infty} \frac{1}{2^q}p_q \right\|_{\infty}$$

is less than ϵ . Then find $\alpha \in \Lambda$ sufficiently large so that $\alpha \subseteq S_{1/2^q}$ or $\alpha \subseteq S_{1/2^q}^c$ for each $q = 1, \dots, N$. (One could first find an α that works for each $S_{1/2^q}$ separately, using the ultrafilter condition (1.4c). Then, find an α that works for all $S_{1/2^q}$, $q = 1, \dots, N$ using the finite intersection property (1.4a).) For any $\beta \in \Lambda$ beyond this α , the first N projections in (1.5) have settled down

to their final value (either 0 or 1) on the net $\{n_\alpha\}_{\alpha \in \Lambda}$. Thus the net of real numbers $\{\varphi(n_\alpha)\}_{\alpha \in \Lambda}$ converges. Since $\{n_\alpha\}_{\alpha \in \Lambda}$ was an arbitrary net from an arbitrary ultrafilter on \mathbb{Z} , this shows that \mathcal{FU} is a well-defined map from the ultrafilters on \mathbb{Z} into $\beta(\mathbb{Z})$. One easily checks that the compositions $\mathcal{UF} \circ \mathcal{FU}$ and $\mathcal{FU} \circ \mathcal{UF}$ are identity maps, and so we have proved:

Proposition 1.6. The map \mathcal{UF} is an isomorphism of the Stone-Čech compactification $\beta(\mathbb{Z})$ with the set of ultrafilters on \mathbb{Z} . Under this map, a point $n_0 \in \beta(\mathbb{Z})$ is taken to the ultrafilter of sets that any net of integers converging to n_0 is eventually in.

3 Definition of the Smooth Functions $\ell^{\infty\infty}(\mathbb{Z})$ (also denoted by $C^\infty(\beta(\mathbb{Z}))$)

Definition 2.1. Define $\ell_c^\infty(\mathbb{Z})$ to be the finite span of projections in $\ell^\infty(\mathbb{Z})$. This is a dense \star -subalgebra of $\ell^\infty(\mathbb{Z})$, and plays an analogous role to $\ell^\infty(\mathbb{Z})$ as $c_c(\mathbb{Z})$, the compact (or finite) support functions on \mathbb{Z} , does to $c_0(\mathbb{Z})$. (The series expansion (1.5) proves the density.)

For $n_0 \in \beta(\mathbb{Z})$, choose a net $\{n_\alpha\}_{\alpha \in \Lambda}$ of integers converging to n_0 . Define the *smooth functions on $\beta(\mathbb{Z})$* , denoted by $\ell^{\infty\infty}(\mathbb{Z})$ or $C^\infty(\beta(\mathbb{Z}))$, to be those

functions φ in $\ell^\infty(\mathbb{Z})$ that satisfy

$$\lim_{\alpha \in \Lambda} n_\alpha^d \left(\varphi(n_\alpha) - \varphi(n_0) \right) = 0 \quad (2.2)$$

for each $d = 0, 1, 2, \dots$ and for each $n_0 \in \beta(\mathbb{Z})$. This set of functions $\ell^{\infty\infty}(\mathbb{Z})$ is a dense \star -subalgebra of $\ell^\infty(\mathbb{Z})$, which contains $\ell_c^\infty(\mathbb{Z})$, and plays an analogous role to $\ell^\infty(\mathbb{Z})$ as $\mathcal{S}(\mathbb{Z})$, the Schwartz functions on \mathbb{Z} , does to $c_0(\mathbb{Z})$. If p is a projection in $\ell^\infty(\mathbb{Z})$, we noticed above that $p(n_0) = 0$ or $p(n_0) = 1$ for any $n_0 \in \beta(\mathbb{Z})$. In either case, the quantity in parentheses in (2.2) eventually becomes 0, so that $\varphi = p$ satisfies (2.2). This shows that $\ell_c^\infty(\mathbb{Z}) \subseteq \ell^{\infty\infty}(\mathbb{Z})$.

Lemma 2.3. The limit (2.2) holds independently of the choice of net $\{n_\alpha\}_{\alpha \in \Lambda}$ converging to n_0 .

Proof: Let $\epsilon > 0$ be given and find a β such that $|n_\alpha^d(\psi(n_\alpha) - \psi(n_0))| < \epsilon$ for $\alpha \geq \beta$. The set $S = \bigcup \{n_\alpha \mid \alpha \geq \beta\}$ is in the ultrafilter associated with n_0 and $|m^d(\psi(m) - \psi(n_0))| < \epsilon$ for $m \in S$. If $\{m_\alpha\}_{\alpha \in \Gamma}$ is another net tending to n_0 , then it is eventually in S . So we have $|m_\alpha^d(\psi(m_\alpha) - \psi(n_0))| < \epsilon$ for $\alpha \geq \gamma$, for some $\gamma \in \Gamma$. **QED**

To see that $\ell^{\infty\infty}(\mathbb{Z})$ is closed under products, let $\varphi, \psi \in \ell^{\infty\infty}(\mathbb{Z})$. Then evaluate the quantity in parentheses in (2.2), namely the difference

$$\varphi(n_\alpha)\psi(n_\alpha) - \varphi(n_0)\psi(n_0) = \left(\varphi(n_\alpha) - \varphi(n_0) \right) \psi(n_\alpha) + \varphi(n_0) \left(\psi(n_\alpha) - \psi(n_0) \right).$$

So the absolute value of the quantity in the limit (2.2) is

$$\left| n_\alpha^d \left(\varphi \psi(n_\alpha) - \varphi \psi(n_0) \right) \right| \leq \left| n_\alpha^d \left(\varphi(n_\alpha) - \varphi(n_0) \right) \right| \left(\|\psi\|_\infty + \|\varphi\|_\infty \right) \left| n_\alpha^d \left(\psi(n_\alpha) - \psi(n_0) \right) \right|$$

Clearly, this tends to zero as n_α tends to n_0 .

Next, we note that $\ell^{\infty\infty}(\mathbb{Z})$ is actually bigger than $\ell_c^\infty(\mathbb{Z})$. For example, any function in $\mathcal{S}(\mathbb{Z})$ satisfies the limit (2.2), so $\ell^{\infty\infty}(\mathbb{Z}) \supseteq \ell_c^\infty(\mathbb{Z}) + \mathcal{S}(\mathbb{Z})$. For $\varphi \in \mathcal{S}(\mathbb{Z})$, note that $\varphi(n_0) = 0$ for any $n_0 \in \beta(\mathbb{Z}) - \mathbb{Z}$. (Any nonprincipal ultrafilter eventually leaves every finite set.) Therefore

$$\left| n_\alpha^d \left(\varphi(n_\alpha) - \varphi(n_0) \right) \right| = \left| n_\alpha^d \varphi(n_\alpha) \right| = 1/n_\alpha^2 \left| n_\alpha^{d+2} \varphi(n_\alpha) \right| \leq 1/n_\alpha^2 \|\varphi\|_{d+2},$$

where $\|\cdot\|_{d+2}$ denotes the $d+2$ th Schwartz seminorm on $\mathcal{S}(\mathbb{Z})$.

Proposition 2.4. The inclusions $\ell_c^\infty(\mathbb{Z}) + \mathcal{S}(\mathbb{Z}) \subseteq \ell^{\infty\infty}(\mathbb{Z}) \subseteq \ell^\infty(\mathbb{Z})$ are proper.

Proof: The function $\frac{1}{n^2+1}$ in $c_0(\mathbb{Z}) - \mathcal{S}(\mathbb{Z})$ does not satisfy (2.2), so the second inclusion is proper. The first inclusion is proper since $e^{i\mathcal{S}(\mathbb{Z})} \subseteq \ell^{\infty\infty}(\mathbb{Z})$. **QED**

Let $\varphi \in \ell^{\infty\infty}(\mathbb{Z})$. We may write φ (in fact any function in $\ell^\infty(\mathbb{Z})$) uniquely in the form

$$\varphi = \sum_{q=1}^{\infty} c_q p_q, \tag{2.5}$$

where the p_q 's are *pairwise disjoint projections*, and the c_q 's are *distinct* constants.

Proposition 2.6. At most finitely many projections in the series (2.5) have infinite support.

Proof: *We assume for a contradiction that there are infinitely many projections in (2.5), each having infinite support.* Note that the coefficients of the infinite projections must have an accumulation point. By dropping to a subsequence, and getting rid of all the finite projections, we may assume that $c_q \rightarrow c_0 \in \mathbb{C}$ as $q \rightarrow \infty$. (Multiply φ by an appropriate projection, and renumber the c_q 's.) Let $S_q \subseteq \mathbb{Z}$ denote the support of the projection p_q . Then each set S_q is infinite. Define a decreasing sequence of infinite subsets of \mathbb{Z} by the disjoint unions

$$U_n = \bigcup_{q \geq n} \left(S_q \cap \left\{ m \in \mathbb{Z} \mid m \geq \frac{1}{|c_q - c_0|} \right\} \right)$$

for $n = 1, 2, \dots$. Since finite intersections of the sets U_n 's are non-empty, there exists an ultrafilter \mathcal{U} for which $U_n \in \mathcal{U}$ for each n . Let $\{n_\alpha\}_{\alpha \in \mathcal{U}}$ be a net of integers that is eventually in this ultrafilter. This net must therefore also eventually be in each of the sets U_n . By construction, the point $n_0 \in \mathcal{FU}(\mathcal{U}) \in \beta(\mathbb{Z})$ that $\{n_\alpha\}_{\alpha \in \mathcal{U}}$ converges to must satisfy $\varphi(n_0) = c_0$.

If $n_\alpha \in S_q \cap \{m \in \mathbb{Z} \mid m \geq \frac{1}{|c_q - c_0|}\}$, then

$$\begin{aligned} \left| n_\alpha^d \left(\psi(n_\alpha) - \psi(n_0) \right) \right| &= \left| n_\alpha^d (c_q - c_0) \right| \quad \text{since } n_\alpha \in S_q \\ &\geq \left| n_\alpha^d \times \frac{1}{n_\alpha} \right| = |n_\alpha^{d-1}| \geq 1, \end{aligned}$$

contradicting the fact that φ must satisfy (2.2) for $d \geq 2$ at the point $n_0 \in \beta(\mathbb{Z})$ we constructed. **QED**

It follows from Proposition 2.6 that for any $\varphi \in \ell^{\infty\infty}(\mathbb{Z})$, we may subtract an element of $\ell_c^\infty(\mathbb{Z})$ to force the expansion (2.5) to have only projections of finite support.

Remark 2.7. Note that $\mathcal{S}(\mathbb{Z})$ is an ideal in $\ell^{\infty\infty}(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$. The closure $\overline{\mathcal{S}(\mathbb{Z})}^{\|\cdot\|_\infty}$ is equal to $c_0(\mathbb{Z})$, so $\mathcal{S}(\mathbb{Z})$ is not dense in $\ell^\infty(\mathbb{Z})$. The algebra $\ell_c^\infty(\mathbb{Z})$, being unital, is *not* an ideal in either algebra $\ell^{\infty\infty}(\mathbb{Z})$ or $\ell^\infty(\mathbb{Z})$, and for the same reason $\ell^{\infty\infty}(\mathbb{Z})$ is not an ideal in $\ell^\infty(\mathbb{Z})$. This is in contrast to the case $c_c(\mathbb{Z}) \subseteq \mathcal{S}(\mathbb{Z}) \subseteq c_0(\mathbb{Z})$, where every algebra is a dense ideal in every algebra above it.

4 References

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